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Pfaff's method (I): The Mills–Robbins–Rumsey determinant [1]

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In honor of Adriano Garsia

Abstract

A new evaluation of

$$\det\left(\binom{i+j+x}{2i-j}\right)_{0\leqslant i,j\leqslant n-1}$$

is provided. The method of proof is inspired by the work of Pfaff. The proof hinges on the summation of many new balanced ${}_5F_4$ hypergeometric series.

1. Introduction

In extensive work on plane partitions and related combinatorial problems [9] (a charming expository account is given in [11]), Mills, Robbins and Rumsey proved the following result (the rising factorial notation

$$(A)_j = A(A+1)\cdots(A+j-1) \tag{1.1}$$

is employed for brevity).

Theorem (Mills et al. [9]). *Let* $\Delta_0(\mu)=2$, *and for* $j>0$

$$\Delta_{2j}(\mu)=\frac{(\mu+2j+2)_j(\frac{1}{2}\mu+2j+\frac{3}{2})_{j-1}}{(j)_j(\frac{1}{2}\mu+j+\frac{3}{2})_{j-1}}, \tag{1.2}$$

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then

$$\det\left(\binom{\mu+i+j}{2i-j}\right)_{0\leqslant i,j\leqslant n-1}=2^{-n}\prod_{k=0}^{n-1}\Delta_{2k}(2\mu). \tag{1.3}$$

Their proof of this result depends on an ingenious determinant factorization theorem [9, Theorem 5, p. 50] together with invocation of an earlier result:

Theorem (Andrews [1]). *For* $j>0$, *if*

$$\Delta_{2j-1}(\mu)=\frac{(\mu+2j)_{j-1}(\frac{1}{2}\mu+2j+\frac{1}{2})_j}{(j)_j(\frac{1}{2}\mu+j+\frac{1}{2})_{j-1}}, \tag{1.4}$$

then

$$\det\left(\delta_{ij}+\binom{\mu+i+j}{j}\right)_{0\leqslant i,j\leqslant n-1}=\prod_{k=0}^{n-1}\Delta_k(\mu). \tag{1.5}$$

Mills, Robbins and Rumsey remark that their proof of (1.3), 'a formula quite simple in appearance, is so far the only one known and makes essential use of [identity (1.5)] whose only known proof [1] is quite complicated'. Also in [11, p. 14, Corollary 2] Robbins again refers to the complexity of the proof of (1.5).

There are two objects in this paper. The primary object is a new proof of (1.3). This proof will be lengthy but less complicated than that of (1.5).

Our secondary object will be an extension of a method [10, p. 51] for proving hypergeometric summations, a method first used by Pfaff to prove an identity often attributed to Saalschutz [12]. This will be required to prove that for nonnegative integers i,

$${}_5F_4\left(\begin{matrix}-2i-1,x+2i+2,x-z+\frac{1}{2},x+i+1,z+i+1;1\\ \frac{x}{2}+\frac{1}{2},\frac{x}{2}+1,2z+2i+2,2x-2z+1\end{matrix}\right)=0, \tag{1.6}$$

where

$${}_{r+1}F_r\left(\begin{matrix}a_0,a_1,\ldots,a_r;t\\ b_1,\ldots,b_r\end{matrix}\right)=\sum_{n=0}^{\infty}\frac{(a_0)_n(a_1)_n\cdots(a_r)_nt^n}{n!(b_1)_n\cdots(b_r)_n}, \tag{1.7}$$

(see [7, p. 8] or [15, p. 40] for background).

It should be emphasized that (1.3) has arisen in a number of significant applications in plane partitions. It appears as the key element in the second proof of the totally symmetric self-complementary plane partitions (TSSCPP) conjecture [5, Section 4] (cf. [10]); the original proof [2] of TSSCPP was also a complicated evaluation of a different determinant due to Stembridge [14]. Also Stembridge [15] has proved the totally symmetric plane partitions conjecture; again (1.3) is required.

The method used here is sufficiently formidable in overall appearance that a few words describing the development of the method are required. The standard methods (e.g. [7, Ch. 4]) available for evaluating series (such as (1.6)) failed for me in this instance. I then began to examine a number of others ${}_5F_4$'s in the hope of finding more tractable results that would provide a clue to (1.6). Instead, to my surprise, I found (via computer experiment) identities (4.2)–(4.21) as well as (6.1)–(6.10) and a number of others. None of these yielded to my attempts to apply classical methods. However, it was easy to see that many of these results were connected together by simple term-by-term comparison (e.g. some of (5.2)–(5.21)). This led to the idea that a simple albeit massive induction might prove everything simultaneously. The hoped-for induction is the subject of Sections 2–6. During preparation I recalled Richard Askey's remark that Pfaff had proved Saalschultz's theorem by induction one hundred years before Saalschultz published his paper. Investigation proved that Pfaff [10] was using exactly the method used here on a much simpler problem. In [3], we shall apply Pfaff's powerful idea to many other summations.

The remainder of this paper is organized as follows. In Section 2, we derive three elementary identities connecting general balanced hypergeometric series. In Section 3, we specialize the results of Section 2 to derive 16 three term relations connecting balanced ${}_5F_4$'s relevant to our project. In Section 4, we state twenty-one identities which we shall prove simultaneously in Section 5; one of these identities is (1.6). Section 6 presents a sampling of hypergeometric corollaries of the previous results. In Section 7, we finally present our new proof of (1.3). We conclude with a discussion of open problems and a brief account of a much more extensive treatment of Pfaff's Method [3].

2. Balanced hypergeometric series

A balanced hypergeometric series is one wherein the sum of the lower parameters (the b_i) is 1 larger than the upper sum (the a_i). Thus, we say

$$ {}_{r+1}F_r\left(\begin{matrix} a_0, a_1, \ldots, a_r; t \\ b_1, \ldots, b_r \end{matrix}\right) \tag{2.1} $$

is *balanced* when $1+\sum_{j=0}^r a_j = \sum_{j=1}^r b_j$.

Theorem 1. *If two balanced hypergeometric series differ only in two parameters and if each difference is* 1 *in absolute value, then these two series together with a third balanced series satisfy a linear three term identity.*

Proof. By symmetry of the parameters we may assume that the difference occurs for either: (1) a_r and b_r, or (2) a_{r-1} and a_r, or (3) b_{r-1} and b_r. The 'balanced' condition halves the number of possible contiguous relations to consider.

In case (1),

$$
\begin{aligned}
&{}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_r;t \\ b_1,\dots,b_r \end{pmatrix} - {}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_{r-1},a_r-1;t \\ b_1,\dots,b_{r-1},b_r-1 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{(a_0)_n\cdots(a_{r-1})_n(a_r)_{n-1}t^n}{n!\cdots(b_{r-1})_n(b_r-1)_{n+1}} \\
&\quad \times((a_r+n-1)(b_r-1)-(a_r-1)(b_r+n-1)) \\
&= (b_r-a_r)\sum_{n=0}^{\infty} \frac{(a_0)_n\cdots(a_{r-1})_n(a_r)_{n-1}nt^n}{n!(b_1)_n\cdots(b_{r-1})_n(b_r-1)_{n+1}} \\
&= \frac{(b_r-a_r)a_0\cdots a_{r-1}t}{b_1\cdots b_{r-1}(b_r-1)b_r}\sum_{n=0}^{\infty} \frac{(a_0+1)_n\cdots(a_{r-1}+1)_n(a_r)_nt^n}{n!(b_1+1)_n(b_{r-1}+1)_n(b_r+1)_n} \\
&= \frac{(b_r-a_r)a_0\cdots a_{r-1}t}{b_1\cdots b_{r-1}(b_r-1)b_r}\,{}_{r+1}F_r\begin{pmatrix} a_0+1,\dots,a_{r-1}+1,a_r;t \\ b_1+1,\dots,b_r+1 \end{pmatrix}. \qquad (2.2)
\end{aligned}
$$

We note that (2.2) is the required three term identity in case (1) because each of the three ${}_{r+1}F_r$'s is balanced if any one is.

In case (2) precisely the same term-by-term subtraction yields

$$
\begin{aligned}
&{}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_r;t \\ b_1,\dots,b_r \end{pmatrix} - {}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_{r-2},a_{r-1}-1,a_r+1;t \\ b_1,\dots,b_r \end{pmatrix} \\
&= \frac{(a_r-a_{r-1})a_0\cdots a_{r-2}t}{b_1\cdots b_r}\,{}_{r+1}F_r\begin{pmatrix} a_0+1,\dots,a_{r-2}+1,a_{r-1},a_r+1;t \\ b_1+1,\dots,b_r+1 \end{pmatrix}, \qquad (2.3)
\end{aligned}
$$

and in case (3), we have

$$
\begin{aligned}
&{}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_r;t \\ b_1,\dots,b_r \end{pmatrix} - {}_{r+1}F_r\begin{pmatrix} a_0,\dots,a_r;t \\ b_1,\dots,b_{r-2},b_{r-1}-1,b_r+1 \end{pmatrix} \\
&= \frac{(b_{r-1}-1-b_r)a_0\cdots a_rt}{b_1\cdots b_{r-2}(b_{r-1}-1)b_{r-1}(b_r+1)b_r} \\
&\quad \times {}_{r+1}F_r\begin{pmatrix} a_0+1,\dots,a_r+1;t \\ b_1+1,\dots,b_{r-2}+1,b_{r-1}+1,b_r+2 \end{pmatrix}. \quad \square \qquad (2.4)
\end{aligned}
$$

3. The 16 three-term relations

Our fundamental interest here is

$$
\begin{aligned}
&H(n,m,a_1,a_2,a_3)=H(n,m;x,z;a_1,a_2,a_3) \\
&= {}_5F_4\begin{pmatrix} -m-n,x+m+n+1+a_1,x-z+\frac{1}{2},x+m+a_2,z+n+1;1 \\ 1+\frac{x}{2},\frac{1+x}{2},2z+m+n+1+a_3,a_1+a_2-a_3+1+2x-2z \end{pmatrix}. \qquad (3.1)
\end{aligned}
$$

The x and z are independent real variables; n and m are non-negative integers; the a_i will be restricted to $-1,0,1$.

We now list 16 specializations of Theorem 1. After each specialization we note which of (2.2)–(2.4) has been utilized.

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1-1,a_2,a_3)$$
$$=\frac{4(m+n)(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(m+n+2z-x+a_3-a_2)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+2x-2z)(a_1+a_2-a_3+2x-2z+1)}$$
$$\times H(n,m-1;x+2,z+1;a_1-1,a_2,a_3)\quad\text{(by (2.2))}, \tag{3.2}$$

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1,a_2-1,a_3)$$
$$=\frac{4(m+n)(x-z+\frac{1}{2})(x+m+n+1+a_1)(z+n+1)(m+2z-x-1+a_3-a_1)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+2x-2z)(a_1+a_2-a_3+2x-2z+1)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2-1,a_3)\quad\text{(by (2.2))}, \tag{3.3}$$

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1,a_2,a_3-1)$$
$$=\frac{4(m+n)(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(-4z+2x-m-n-2a_3+a_1+a_2+1)}{(x+1)(x+2)(2z+m+n+a_3)(2z+m+n+a_3+1)(a_1+a_2-a_3+1+2x-2z)(a_1+a_2-a_3+2+2x-2z)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2,a_3)\quad\text{(by (2.4))}, \tag{3.4}$$

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1-1,a_2+1,a_3)$$
$$=\frac{4(m+n)(x-z+\frac{1}{2})(z+n+1)(a_1-a_2+n)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n,m-1;x+2,z+1;a_1-1,a_2,a_3)\quad\text{(by (2.3))}, \tag{3.5}$$

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1-1,a_2,a_3-1)$$
$$=\frac{4(m+n)(x+m+a_2)(x-z+\frac{1}{2})(z+n+1)(x-2z+a_1-a_3)}{(x+1)(x+2)(a_1+a_2-a_3+1+2x-2z)(2z+m+n+a_3)(2z+m+n+a_3+1)}$$
$$\times H(n,m-1;x+2,z+1;a_1-1,a_2,a_3)\quad\text{(by (2.2))}, \tag{3.6}$$

$$H(n,m,a_1,a_2,a_3)-H(n,m,a_1,a_2-1,a_3-1)$$
$$=\frac{4(m+n)(x+m+n+1+a_1)(x-z+\frac{1}{2})(z+n+1)(-2z+x-n-a_3+a_2-1)}{(x+1)(x+2)(a_1+a_2-a_3+1+2x-2z)(2z+m+n+a_3)(2z+m+n+a_3+1)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2-1,a_3)\quad\text{(by (2.2))}, \tag{3.7}$$

$$H(n,m,a_1,a_2,a_3)-H(n-1,m,a_1+1,a_2,a_3+1)$$
$$=\frac{-4(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(m+2n+z)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n-1,m;x+2,z+1;a_1,a_2-1,a_3)\quad\text{(by (2.2))}, \tag{3.8}$$

$$H(n,m,a_1,a_2,a_3) - H(n,m-1,a_1+1,a_2,a_3+1)$$
$$= \frac{-4(x+m+n+1+a_1)(x-z+\frac{1}{2})(z+n+1)(2m+n+x+a_2-1)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2-1,a_3) \quad \text{(by (2.3))}, \tag{3.9}$$

$$H(n,m,a_1,a_2,a_3) - H(n,m-1,a_1,a_2+1,a_3+1)$$
$$= \frac{-4(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(2m+2n+x+a_1)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n,m-1;x+2,z+1;a_1-1,a_2,a_3) \quad \text{(by (2.3))}, \tag{3.10}$$

$$H(n,m,a_1,a_2,a_3) - H(n-1,m+1,a_1,a_2,a_3)$$
$$= \frac{4(m+n)(x+m+n+1+a_1)(x-z+\frac{1}{2})(z+n-x-m-a_2)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n-1,m;x+2,z+1;a_1,a_2-1,a_3) \quad \text{(by (2.3))}, \tag{3.11}$$

$$H(n,m,a_1,a_2,a_3) - H(n-1,m+1,a_1,a_2-1,a_3)$$
$$= \frac{4(m+n)(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(2x-3z-n+a_1+a_2-a_3)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+2x-2z)(a_1+a_2-a_3+2x-2z+1)}$$
$$\times H(n-1,m;x+2,z+1;a_1,a_2-1,a_3) \quad \text{(by (2.3))}, \tag{3.12}$$

$$H(n,m,a_1,a_2,a_3) - H(n,m-1,a_1+1,a_2+1,a_3+2)$$
$$= \frac{-4(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(2z+2m+2n+1+a_3)}{(x+1)(x+2)(a_1+a_2-a_3+1+2x-2z)(2z+m+n+1+a_3)(2z+m+n+2+a_3)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2,a_3+1) \quad \text{(by (2.2))}, \tag{3.13}$$

$$H(n,m,a_1,a_2,a_3) - H(n,m-1,a_1+1,a_2+1,a_3+1)$$
$$= \frac{-4(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(x+m+n+1+a_1)(m+n+a_1+a_2-a_3+1+2x-2z)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)(a_1+a_2-a_3+2+2x-2z)}$$
$$\times H(n,m-1;x+2,z+1;a_1,a_2,a_3) \quad \text{(by (2.2))}, \tag{3.14}$$

$$H(n,m,a_1,a_2,a_3) - H(n-1,m+1,a_1+1,a_2-1,a_3)$$
$$= \frac{-4(m+n)(x-z+\frac{1}{2})(x+m+a_2)(x-z+m+1+a_1)}{(x+1)(x+2)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$
$$\times H(n-1,m;x+2,z+1;a_1,a_2-1,a_3) \quad \text{(by (2.3))}, \tag{3.15}$$

$$H(n,m,a_1,a_2,a_3) - H(n-1,m+1,a_1,a_2-1,a_3-1)$$

$$= \frac{-4(m+n)(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(z+m+a_3)}{(x+1)(x+2)(a_1+a_2-a_3+1+2x-2z)(2z+m+n+a_3)(2z+m+n+a_3+1)}$$

$$\times H(n-1,m;x+2,z+1;a_1,a_2-1,a_3) \quad \text{(by (2.2))}, \tag{3.16}$$

$$H(n,m;x,z;a_1,a_2,a_3) - H(n-1,m;x+1,z+1;a_1,a_2-1,a_3-1)$$

$$= \frac{-4(x+m+n+1+a_1)(x-z+\frac{1}{2})(x+m+a_2)(z+n+1)(2m+2n+x+1)}{(x+1)(x+2)(x+3)(2z+m+n+1+a_3)(a_1+a_2-a_3+1+2x-2z)}$$

$$\times H(n-1,m;x+3,z+2;a_1-1,a_2-2,a_3-2) \quad \text{(by (2.2))}. \tag{3.17}$$

4. The 20 identities

To formulate our identities succinctly we define

$$P_n = P_n(x,z) = \frac{(\frac{1}{2})_n(2z-x)_{2n}}{(x+1)_n(1+x-z)_n(z+n+\frac{1}{2})_n}. \tag{4.1}$$

The following 20 assertions will be proved in Section 5.

$$H(n,n+1,0,0,0) = 0, \tag{4.2}$$

$$H(n,n,0,0,0) = \frac{(x+n)(2z-x+2n)P_n}{(x+2n)(2z-x+n)}, \tag{4.3}$$

$$H(n-1,n+1,0,0,0) = P_n, \tag{4.4}$$

$$H(n,n+1,0,-1,0)$$

$$= -\frac{(x+n)(x+n+1)(z-x-n-1)(2z+4n+3)P_{n+1}}{2(x+2n)(x+2n+1)(z-x)(2z-x+n+1)}, \tag{4.5}$$

$$H(n,n,0,-1,0)$$

$$= \frac{(x+n-2)(x+n-1)(x+n)(z-x-n)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n-2)(x+2n-1)(x+2n)(z-x)(2z-x+n)(2z-x+n+1)}, \tag{4.6}$$

$$H(n-1,n+1,0,-1,0) = \frac{(x+n)(z-x-n)}{(x+2n)(z-x)}P_n, \tag{4.7}$$

$$H(n,n+1,1,0,1)$$

$$= \frac{-(2n+1)(2z+2n+1)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n+2)(2z+4n+1)(2z+4n+3)(2z-x+n+1)}, \tag{4.8}$$

$$H(n,n,1,0,1)$$

$$= \frac{(x+n)(x+3n+1)(2z+2n+1)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n)(x+2n+1)(2z+4n+1)(2z-x+n)(2z-x+n+1)}, \tag{4.9}$$

$$H(n-1,n+1,1,0,1)=\frac{(z+3n+1)(2z+2n+1)}{(z+n+1)(2z+4n+1)}P_n, \tag{4.10}$$

$$H(n,n+1,0,0,1)=\frac{-(x+n+1)(z-x-n-1)}{(z-x)(2z-x)}P_{n+1}, \tag{4.11}$$

$$H(n,n,0,0,1)$$
$$=\frac{(x+n)(z-x-n)(2z+2n+1)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n)(z-x)(2z+4n+1)(2z-x)(2z-x+n+1)}, \tag{4.12}$$

$$H(n,n+1,-1,0,0)=\frac{(x+n+1)(2z+4n+3)(z-x-n-1)P_{n+1}}{2(x+4n+2)(z-x)(2z-x)}, \tag{4.13}$$

$$H(n,n,-1,0,0)=\frac{(x+n)(z-x-n)(2z-x+2n)P_n}{(x+4n)(z-x)(2z-x)}, \tag{4.14}$$

$$H(n,n+1,0,1,1)=\frac{z-x-n-1}{2z-x}P_{n+1}, \tag{4.15}$$

$$H(n,n,0,1,1)=\frac{(2z+2n+1)(2z-x+2n)P_n}{(2z+4n+1)(2z-x)}, \tag{4.16}$$

$$H(n,n+1,-1,-1,-1)$$
$$=\frac{(2n+1)(x+n)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n)(x+4n+2)(z-x)(2z+4n+1)}, \tag{4.17}$$

$$H(n-1,n+1,-1,-1,-1)=\frac{(x+n)(z-x-n)}{(x+4n)(z-x)}P_n, \tag{4.18}$$

$$H(n,n+1,-1,-1,0)$$
$$=\frac{(4n+2)(x+n)(z-x-n)(2z+2n+1)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n)(x+4n+2)(z-x)(2z+4n+1)(2z-2x+1)(2z-x)}, \tag{4.19}$$

$$H(n,n+1,0,-1,-1)=\frac{(2n+1)(x+n)(2z-x+2n)(2z-x+2n+1)P_n}{(x+2n)(x+2n+1)(2z+4n+1)(2z-x+n)}, \tag{4.20}$$

$$H(n,n+1,-1,0,-1)=\frac{(2n+1)(2z-x+2n)P_n}{(x+4n+2)(2z+4n+1)}. \tag{4.21}$$

In the next section, we shall provide identities (4.2)–(4.21).

5. The proof

Theorem 2. *Identities* (4.2)–(4.21) *are valid for* $n \geqslant 0$.

Proof. We proceed with the world's most cumbersome mathematical induction ever.

It is easy to show that all 20 identities are valid for $n=0$. Indeed many are valid for $n=-1$ provided we make the natural extension of P_n for $n=-1$, namely

$$P_{-1}=P_{-1}(n,x,z)=\frac{x(z-x)(2z-3)}{(2z-x-2)(2z-x-1)}. \tag{5.1}$$

(We have checked (4.2)–(4.21) on MACSYMA and/or AXIOM for $n=0,1,2,3$).

The following is a list of 20 recurrences wherein each $H(n,m,a_1,a_2,a_3)$ listed in (4.2)–(4.21) appears on the left-hand side. (Note that in each instance the m parameter is always dependent on n [indeed $m=n$, $n-1$ or $n+1$]; thus the induction is on n as it appears in each of (4.2)–(4.21) not just on the first variable in $H(n,m,a_1,a_2,a_3)$.) In the first six identities, the entries on the right-hand side are all H's with $n-1$ replacing n. In the remaining 14 identities, all the entries on the right are H's that either have $n-1$ replacing n or have appeared previously on the left-hand side in this list of recurrences. Consequently, (5.2)–(5.21) together with the case $n=0$ uniquely define our 20 H's.

Hence once (5.2)–(5.21) are established for the H's, all that remains is a verification that the right-hand sides of (4.2)–(4.21) satisfy (5.2)–(5.21).

$$\begin{aligned} H(n,n,-1,0,0)&=H(n-1,n,0,0,1)\\ &\quad+\frac{2(x+2n)(x-z+\frac{1}{2})(x+n)(z+3n)}{(x+1)(x+2)(2z+2n+1)(z-x)}\\ &\quad\times H(n-1,n;x+2,z+1;-1,-1,0)\quad\text{(by (3.8))}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} H(n,n,0,0,0)&=H(n-1,n,1,0,1)\\ &\quad-\frac{2(x+2n+1)(x+n)(z+3n)}{(x+1)(x+2)(2z+2n+1)}\\ &\quad\times H(n-1,n;x+2,z+1;0,-1,0)\quad\text{(by (3.8))}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} H(n-1,n+1,0,0,0)&=H(n-1,n,1,0,1)\\ &\quad-\frac{2(x+2n+1)(z+n)(x+3n)}{(x+1)(x+2)(2z+2n+1)}\\ &\quad\times H(n-1,n;x+2,z+1;0,-1,0)\quad\text{(by (3.9))}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} H(n-1,n+1,0,-1,0)&=H(n-1,n,0,0,1)\\ &\quad+\frac{2(x-z+\frac{1}{2})(x+n)(z+n)(x+4n)}{(x+1)(x+2)(2z+2n+1)(z-x)}\\ &\quad\times H(n-1,n;x+2,z+1;-1,-1,0)\quad\text{(by (3.10))}, \end{aligned} \tag{5.5}$$

$$H(n-1,n+1,-1,-1,-1)=H(n-1,n,0,0,0)-\frac{(x+n)(x+2n)(n+x-z)}{(x+1)(x+2)(z-x)}\times H(n-1,n;x+2,z+1;-1,-1,-1)\quad\text{(by (3.14))},\tag{5.6}$$

$$H(n,n+1,-1,0,-1)=H(n-1,n+1,0,0,0)-\frac{2(x+2n+1)(x+n+1)(z+3n+1)}{(x+1)(x+2)(2z+2n+1)}\times H(n-1,n+1;x+2,z+1;-1,-1,-1)\quad\text{(by (3.8))},\tag{5.7}$$

$$H(n,n+1,0,1,1)=H(n-1,n+1;x+1,z+1;0,0,0)-\frac{2(x+2n+2)(x+n+2)(z+n+1)(x+4n+3)}{(x+1)(x+2)(x+3)(2z+2n+3)}\times H(n-1,n+1;x+3,z+2;-1,-1,-1)\quad\text{(by (3.17))},\tag{5.8}$$

$$H(n,n+1,-1,0,0)=H(n,n+1,-1,0,-1)+\frac{(n+\frac{1}{2})(x+2n+1)(x+n+1)(4z+2n+1-2x)}{(x+1)(x+2)(2z+2n+1)(z-x)}\times H(n,n;x+2,z+1;-1,0,0)\quad\text{(by (3.4))},\tag{5.9}$$

$$H(n,n,0,1,1)=H(n-1,n+1,0,0,0)-\frac{2n(x+2n+1)(x+n+1)}{(x+1)(x+2)(2z+2n+1)}\times H(n-1,n;x+2,z+1;0,0,1)\quad\text{(by (3.16))},\tag{5.10}$$

$$H(n-1,n+1,1,0,1)=H(n,n,0,1,1)+\frac{2n(x+n+1)(x-z+n+1)}{(x+1)(x+2)(z+n+1)}\times H(n-1,n;x+2,z+1;0,0,1)\quad\text{(by (3.15) with } n=m,\ a_1=0,\ a_2=a_3=1),\tag{5.11}$$

$$H(n,n+1,0,0,0)=H(n-1,n+1,1,0,1)-\frac{(x+2n+2)(x+n+1)(z+3n+1)}{(x+1)(x+2)(z+n+1)}\times H(n-1,n+1;x+2,z+1;0,-1,0)\quad\text{(by (3.8))},\tag{5.12}$$

$$H(n,n,0,0,1)=\frac{(2z+2n+1)(2z-x+n)}{2(z-x)(2z-x+n+1)}H(n,n,0,0,0) - \frac{(x+n)(4z-2x+2n+1)}{2(z-x)(2z-x+n+1)}H(n,n,0,1,1). \tag{5.13}$$

This result requires more effort in proof than the previous ones. By (3.3)

$$H(n,n,0,1,1)-H(n,n,0,0,1) = \frac{n(x+2n+1)(x-2z-n)}{(x+1)(x+2)(x-z)}H(n,n-1;x+2,z+1;0,0,1),$$

and by (3.4)

$$H(n,n,0,0,1)-H(n,n,0,0,0) = -\frac{n(x+2n+1)(x+n)(4z-2x+2n+1)}{(x+1)(x+2)(2z+2n+1)(x-z)}H(n,n-1;x+2,z+1;0,0,1).$$

Eliminating $H(n,n-1;x+2,z+1;0,0,1)$ from these last two equations, we obtain upon simplification (5.13).

$$H(n,n,1,0,1)=\frac{(2n+1)(z-x)}{(x+2n+1)(2z-x+n)}H(n,n,0,0,1) + \frac{(x+n)(2z-x+2n+1)}{(x+2n+1)(2z-x+n)}H(n,n,0,1,1). \tag{5.14}$$

The derivation of (5.14) is similar to that of (5.13). By (3.2)

$$H(n,n,1,0,1)-H(n,n,0,0,1) = \frac{n(x+n)(2z+2n-x+1)}{(x+1)(x+2)(x-z)}H(n,n-1;x+2,z+1;0,0,1)$$

and by (3.3)

$$H(n,n,0,1,1)-H(n,n,0,0,1) = \frac{n(x+2n+1)(2z-x+n)}{(x+1)(x+2)(z-x)}H(n,n-1;x+2,z+1;0,0,1).$$

Eliminating $H(n,n-1;x+2,z+1;0,0,1)$ from these last two equations, we obtain upon simplification (5.14).

$$H(n,n,0,-1,0)=-\frac{x(x-1)}{(x+2n)(x+3n-1)}\{H(n,n+1;x-2,z-1;0,0,0) - H(n,n;x-2,z-1;1,0,1)\}. \tag{5.15}$$

Again we require two auxiliary equations to prove (5.15). Namely by (3.8)

$$H(n+1,n,0,0,0)-H(n,n,1,0,1)$$
$$=-\frac{(x+2n+2)(x+n)(z+3n+2)}{(x+1)(x+2)(z+n+1)}H(n,n;x+2,z+1;0,-1,0),$$

and by (3.11)

$$H(n+1,n,0,0,0)-H(n,n+1,0,0,0)$$
$$=\frac{(2n+1)(x+2n+2)(z-x+1)}{(x+1)(x+2)(z+n+1)}H(n,n;x+2,z+1;0,-1,0).$$

If we now subtract the second of these equations from the first and then replace x by $x-2$ and z by $z-1$, we obtain (5.15).

$$\begin{aligned}H(n,n+1,0,0,1)=H(n,n+1,0,0,0)&\\+\frac{(2n+1)(x+2n+2)(x+n+1)(2z-x+n+1)}{(x+1)(x+2)(2z+2n+3)(z-x)}&\\\times H(n,n;x+2,z+1;0,0,1)\quad\text{(by (3.4))},&\end{aligned}\tag{5.16}$$

$$\begin{aligned}&H(n,n+1,0,-1,0)\\&=H(n,n+1,0,0,0)+\frac{(n+\frac{1}{2})(x+2n+2)(x-2z-n)}{(x+1)(x+2)(x-z)}\\&\quad\times H(n,n;x+2,z+1;0,-1,0)\quad\text{(by (3.3) with } a_1=a_2=a_3=0),\end{aligned}\tag{5.17}$$

$$\begin{aligned}&H(n,n+1,0,-1,-1)\\&=H(n,n+1,0,-1,0)+\frac{(n+\frac{1}{2})(x+2n+2)(x+n)(4z-2x+2n+1)}{(x+1)(x+2)(2z+2n+1)(x-z)}\\&\quad\times H(n,n;x+2,z+1;0,-1,0)\quad\text{(by (3.4) with } a_1=a_3=0,\ a_2=-1),\end{aligned}\tag{5.18}$$

$$\begin{aligned}&H(n,n+1,1,0,1)\\&=H(n,n+1,0,1,1)+\frac{2(2n+1)(z+n+1)(n+1)}{(x+1)(x+2)(2z+2n+3)}\\&\quad\times H(n,n;x+2,z+1;0,0,1)\quad\text{(by (3.5))},\end{aligned}\tag{5.19}$$

$$\begin{aligned}&H(n,n+1,-1,-1,0)\\&=\frac{-(x+2n+1)(2z-x+n+1)}{(n+1)(2x-2z-1)}H(n,n+1,0,-1,0)\\&\quad+\frac{(x+n)(2z-x+2n+2)}{(n+1)(2x-2z-1)}H(n,n+1,-1,0,0).\end{aligned}\tag{5.20}$$

As with (5.13), (5.14) and (5.15), we require two auxiliary equations. First by (3.2)

$$H(n,n+1,0,-1,0)-H(n,n+1,-1,-1,0)$$
$$=\frac{-(2n+1)(x-z+\frac{1}{2})(x+n)(x-2z-2n-2)}{(x+1)(x+2)(2x-2z-1)(x-z)}$$
$$\times H(n,n;x+2,z+1;-1,-1,0),$$

and by (3.3)

$$H(n,n+1,-1,0,0)-H(n,n+1,-1,-1,0)$$
$$=\frac{-(2n+1)(x-z+\frac{1}{2})(x+2n+1)(x-2z-n-1)}{(x+1)(x+2)(2x-2z-1)(x-z)}$$
$$\times H(n,n;x+2,z+1;-1,-1,0).$$

Eliminating $H(n,n;x+2,z+1;-1,-1,0)$ from these last two equations, we obtain (5.20) upon simplification.

Finally,

$$H(n,n+1,-1,-1,-1)=\frac{(x+2n+1)(2z-x+n)}{2(n+1)(z-x)}H(n,n+1,0,-1,-1)$$
$$-\frac{(x+n)(2z-x+2n+1)}{2(n+1)(z-x)}H(n,n+1,-1,0,-1). \tag{5.21}$$

To establish (5.21) we note that by (3.2)

$$H(n,n+1,0,-1,-1)-H(n,n+1,-1,-1,-1)$$
$$=\frac{(2n+1)(x+n)(z+n+1)(x-2z-2n-1)}{(x+1)(x+2)(2z+2n+1)(z-x)}$$
$$\times H(n,n;x+2,z+1;-1,-1,-1),$$

and by (3.3)

$$H(n,n+1,-1,0,-1)-H(n,n+1,-1,-1,-1)$$
$$=\frac{(2n+1)(x+2n+1)(z+n+1)(x-2z-n)}{(x+1)(x+2)(2z+2n+1)(z-x)}$$
$$\times H(n,n;x+2,z+1;-1,-1,-1).$$

Eliminating $H(n,n;x+2,z+1;-1,-1,-1)$ from these last two equations, we obtain (5.21) upon simplification.

All that remains now, in light of our remarks preceding (5.2), is to verify that the finite products on the right-hand sides of (4.2)–(4.21) satisfy the same initial conditions at $n=0$ and the twenty defining recurrences (5.2)–(5.21). Since each right-hand side is a finite product of linear terms, the required verifications are just exercises in college algebra. We have carried each one out on MACSYMA. Since each resembles the rest, we illustrate these calculations with one example. Let us represent the right-hand side of each of (4.2)–(4.21) by the notation of the corresponding left-hand side with 'h' replacing 'H'.

As a typical example, to verify (5.15) for the h's we must show that

$$\begin{aligned}&\frac{(x+n-2)(x+n-1)(x+n)(z-x-n)(2z-x+2n)(2z-x+2n+1)P_n(x,z)}{(x+2n-1)(x+2n-1)(x+2n)(z-x)(2z-x+n)(2z-x+n+1)}\\&=\frac{-x(x-1)}{(x+2n)(x+3n-1)}\\&\quad\times\left\{0-\frac{(x+n-2)(x+3n-1)(2z+2n-1)(2z-x+2n)(2z-x+2n+1)P_n(x-2,z-1)}{(x+2n-2)(x+2n-1)(2z+4n-1)(2z-x+n)(2z-x+n+1)}\right\},\end{aligned} \tag{5.22}$$

and this is a routine calculation.

Hence Theorem 2 is proved. □

6. Related identities

Our point in this brief section is to emphasize that there are numerous closely related identities that can be derived from Theorem 2 and the identities of Section 3. We list ten below; however, there are many others:

$$H(n-1,n+1,0,1,0)=P_{n+1}, \tag{6.1}$$

$$H(n-1,n+1,0,0,-1)=P_{n+1}, \tag{6.2}$$

$$H(n-1,n+1,-1,0,-1)=\frac{(x+n)P_{n+1}}{(x+4n+4)}, \tag{6.3}$$

$$H(n,n,-1,0,-1)=\frac{(x+n)(z+3n)}{(x+4n)(z+n)}P_n, \tag{6.4}$$

$$H(n,n,0,1,0)=P_n, \tag{6.5}$$

$$H(n,n,1,1,1)=\frac{(2z+2n+1)(2z-x+2n)}{(2z+4n+1)(2z-x+n)}P_n, \tag{6.6}$$

$$H(n,n+1,1,1,1)=P_{n+1}, \tag{6.7}$$

$$H(n,n+1,1,0,0)=\frac{(x+n)(2z+4n+3)P_{n+1}}{2(x+2n+2)(2z-x+n)}, \tag{6.8}$$

$$H(n,n+1,0,1,0)=\frac{(2z+4n+3)P_{n+1}}{2(2z-x+2n+1)}, \tag{6.9}$$

$$H(n,n+1,-1,1,0)=\frac{n(z-x-n-1)(2z+4n+3)P_{n+1}}{(x+4n+2)(2z-x)(2z-x+2n+1)}. \tag{6.10}$$

7. The proof of (1.3)

We define three $n\times n$ matrices

$$M_n(\mu)=\left(\binom{\mu+i+j}{2i-j}\right)_{0\leqslant i,j\leqslant n-1}, \tag{7.1}$$

$$E_n(\mu)=(e_{i,j}(\mu))_{0\leqslant i,j\leqslant n-1}, \tag{7.2}$$

with

$$e_{i,j}(\mu)=\begin{cases}0 & \text{if } i>j\\ \frac{(-1)^{j-i}(i)_{2(j-i)}(2\mu+2j+i+2)_{j-i}}{4^{j-i}(j-i)!(\mu+i+1)_{j-i}(\mu+j+i+\frac{1}{2})_{j-i}}.\end{cases} \tag{7.3}$$

Note that $E_n(\mu)$ is an upper triangular matrix with ones on the main diagonal.

$$L_n(\mu)=(\ell_{i,j}(\mu))_{0\leqslant i,j\leqslant n-1}=M_n(\mu)E_n(\mu). \tag{7.4}$$

If we can show that $L_n(\mu)$ is lower triangular with main diagonal $\frac{1}{2}\Delta_0(2\mu), \frac{1}{2}\Delta_2(2\mu), \frac{1}{2}\Delta_4(2\mu),\dots,\frac{1}{2}\Delta_{2n-2}(2\mu)$, then we have immediately (1.3) just by taking determinants of the matrix identity (7.4).

Now for $i=j=0$, $\ell_{0,0}(\mu)=1=\frac{1}{2}\Delta_0(2\mu)$, and for $0\leqslant i\leqslant j$ with $j>0$, we have

$$\begin{aligned}
\ell_{i,j}(\mu) &= \sum_{k=0}^{j}\binom{i+k+\mu}{2i-k}e_{k,j}(\mu)\\
&= \sum_{k=1}^{j}\binom{i+k+\mu}{2i-k}e_{k,j}(\mu) \quad (\text{since } e_{0,j}(\mu)=0 \text{ for } j>0)\\
&= \sum_{k=0}^{j-1}\binom{i+k+\mu+1}{2i-k-1}e_{k+1,j}(\mu)
\end{aligned}$$

$$=\sum_{k=0}^{j-1}\binom{i+\mu+1}{2i-1}\frac{(i+\mu+2)_k(-2i+1)_k(-1)^k}{(\mu-i+3)_{2k}}$$

$$\times\frac{(-1)^{j-1}(2j-2)!(2\mu+2j+3)_{j-1}}{(j-1)!4^{j-1}(\mu+2)_{j-1}(\mu+j+\frac{3}{2})_{j-1}}\frac{(-1)^k(-j+1)_k(\mu+2)_k(\mu+j+\frac{3}{2})_k}{4^{-k}k!(-2j+2)_k(2\mu+2j+3)_k}$$

$$=\binom{i+\mu+1}{2i-1}\frac{(-1)^{j-1}(2j-2)!(2\mu+2j+3)_{j-1}}{(j-1)!4^{j-1}(\mu+2)_{j-1}(\mu+j+\frac{3}{2})_{j-1}}$$

$$\times\,{}_5F_4\left(\begin{matrix}-2i+1,i+\mu+2,-j+1,\ \mu+2,\mu+j+\frac{3}{2};1\\ \frac{\mu-i}{2}+\frac{3}{2},\frac{\mu-i}{2}+2,\ -2j+2,2\mu+2j+3\end{matrix}\right)$$

(see next paragraph if $i=j$)

$$= \binom{i+\mu+1}{2i-1} \frac{(-1)^{j-1}(2j-2)!(2\mu+2j+3)_{j-1}}{(j-1)!4^{j-1}(\mu+2)_{j-1}(\mu+j+\frac{3}{2})_{j-1}}$$

$$\times H(i-1,i;\mu-i+2,-j-i+1;0,0,0)$$

$$= 0 \tag{7.5}$$

if $i<j$ by Theorem 2, Eq. (4.2).

If $i=j$, then not all terms in the above ${}_5F_4$ past the jth term vanish. Inspection reveals that when k the index of summation $=2i-1=2j-1$ then the $2j-1$ summand has a removable singularity. To reveal exactly what results we must argue by continuity. Consequently by Theorem 2, Eq. (4.2)

$$0 = \lim_{z\to j-1} H(j-1,j;\mu-j+2,-j-z;0,0,0)$$

$$= \lim_{z\to j-1}\left\{\sum_{k=0}^{j-1}\frac{(-2j+1)_k(j+\mu+2)_k(-z)_k(\mu+2)_k(\mu+z+\frac{5}{2})_k}{k!(\frac{\mu-j+3}{2})_k(\frac{\mu-j+4}{2})_k(-2z)_k(2\mu+2z+5)_k}\right.$$

$$\left.+\sum_{k=j}^{2j-1}\frac{(-2j+1)_k(j+\mu+2)_k(-z)_k(\mu+2)_k(\mu+z+\frac{5}{2})_k}{k!(\frac{\mu-j+3}{2})_k(\frac{\mu-j+4}{2})_k(-2z)_k(2\mu+2z+5)_k}\right\}$$

$$=\ell_{j,j}(\mu)\frac{(j-1)!4^{j-1}(\mu+2)_{j-1}(\mu+j+\frac{3}{2})_{j-1}(2j-1)!(\mu-j+2)!}{(-1)^{j-1}(2j-2)!(2\mu+2j+3)_{j-1}(j+\mu+1)!}$$

$$+\frac{(-2j+1)_{2j-1}(j+\mu+2)_{2j-1}(\mu+2)_{2j-1}(\mu+j+\frac{3}{2})_{2j-1}}{(2j-1)!(\frac{\mu-j+3}{2})_{2j-1}(\frac{\mu-j+4}{2})_{2j-1}(2\mu+2j+3)_{2j-1}}$$

$$\times\lim_{z\to j-1}\frac{(-z)_{2j-1}}{(-2z)_{2j-1}}$$

$$=\ell_{j,j}(\mu)\frac{(j-1)!4^{j-1}(\mu+2)_{j-1}(\mu+j+\frac{3}{2})_{j-1}(2j-1)!}{(-1)^{j-1}(2j-2)!(2\mu+2j+3)_{j-1}(\mu+j+3)_{2j-1}}$$

$$-\frac{(2j+2\mu+3)_{4j-2}(\mu+2)_{2j-1}(-1)^{j-1}(j-1)!^2}{(\mu-j+3)_{4j-2}(2\mu+2j+3)_{2j-1}2(2j-2)!}.$$

Consequently,

$$\ell_{j,j}(\mu)=\frac{(2\mu+4j+2)_{2j-1}(\mu+j+1)_j(2\mu+2j+3)_{j-1}}{2(j)_j(\mu+j+2)_{2j-1}(\mu+j+\frac{3}{2})_{j-1}4^{j-1}}$$

$$=\frac{2^{2j-2}(\mu+2j+1)_j(\mu+2j+\frac{3}{2})_{j-1}(\mu+j+1)_j(2\mu+2j+3)_{j-1}}{(j)_j(\mu+j+2)_{2j-1}(\mu+j+\frac{3}{2})_{j-1}4^{j-1}}$$

$$=\frac{1}{2}\frac{(\mu+2j+\frac{3}{2})_{j-1}(2\mu+2j+2)_j}{(j)_j(\mu+j+\frac{3}{2})_{j-1}}$$

$$=\tfrac{1}{2}\Delta_{2j}(2\mu),$$

and this concludes the proof of (1.3).

8. Conclusion

While the main and original object of this paper was a new proof of the Mills–Robbins–Rumsey determinant evaluation (1.3), we also had some related projects in mind.

First, the method of Pfaff supplies illuminating proofs of many other hypergeometric summations. In [3] we hope to provide an account of numerous applications of this long neglected but powerful method.

Third, we must note that the W–Z method [16,17], which seems the obvious choice for proving (1.6), turns out to be somewhat ill-suited for this project in full generality. So far neither Zeilberger nor I have been able to determine the WZ recurrence for the ${}_5F_4$ in (4.6). Zeilberger has been able to modify the WZ method to include Pfaff-type variable shifts, and he has found a third order recurrence proof for the ${}_5F_4$ in (4.2) that occupies about 17 pages of printout. In [4], we hope to contrast the W–Z method and Pfaff's method more generally. Quite recently, Wilf and Petkovsek have used the WZ method to prove (1.3), and indeed each instance of (1.6) necessary to do (7.5). The secret of their success lies in taking $z=-j-i+1$ as an integer instead of an arbitrary real (or complex) variable. This allows them to use $j-1$ as the integer terminating the ${}_5F_4$ in (7.5) rather than $(2i-1)$. They pay a small price for this change in that the $i=j$ case requires an entirely separate (but none too onerous) evaluation. *However*, they make a great gain in that the WZ method now handles the problem with felicity. The necessary recurrences are all of second order with quite manageable coefficients. It is still a mystery why the WZ method becomes so heavy when Z is an arbitrary real variable in (1.6). We hope to say more about this in [4] also.

Finally, Dennis Stanton has found an alternative proof of (1.6) using methods he and Ira Gessel developed [8,9]. Stanton and I will present a full account of this work and its applications in [6].

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